3. Dynamic Moral Hazard

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Basic Readings

Textbooks:
- Bolton and Dewatripont (2005), Chapter 10
- Schmidt (1995), Chapter 3

Papers:
- Hellwig and Schmidt (2002)
Main Questions

The literature on dynamic moral hazard problems has focused on three questions:

1. Is it possible to achieve a more efficient allocation if principal and agent interact more than once?
2. What are the advantages of writing a long-term contract rather than a sequence of short-term contracts?
3. Do optimal contracts in a repeated relationship have more structure than optimal contracts in a one-shot relationship?
An Infinitely Repeated Principal-Agent Problem

Suppose the agent is maximizing either

\[
\sum_{t=1}^{T} \frac{V(w_t^t) - G(a_t)}{T}, \quad T \to \infty
\]

or

\[
(1 - \delta) \sum_{t=1}^{\infty} \delta^t [V(w_t^t) - G(a_t)] , \quad \delta \to 1 .
\]

The principal maximizes either his average profit or the discounted NPV of all future profits, but he is assumed to be risk neutral.
An Infinitely Repeated Principal-Agent Problem

Proposition 3.1

If $T \to \infty$ ($\delta \to 1$) then the first best can be approximated arbitrarily closely.

This result is due to Radner (1981) and Rubinstein (1981), and the intuition is fairly simple:

The parties write a long-term contract that gives a fixed wage to the agent unless the distribution of outcomes generated by his actions deviates “too much” from the distribution that would have been generated had the agent always taken the first best action. The Law of Large Numbers implies that any systematic shirking of the agent can be detected with a probability arbitrarily close to one. In this case the agent is punished very harshly.
This result is also familiar from the literature on the Folk Theorem in repeated games with public monitoring.

However, the agent has to be very patient for this result to hold.

In the chapter on “Relational Contracts” we consider the question what the parties can implement for a discount factor that is bounded away from 1.
The Radner-Rubinstein result seems to suggest that long-term contracts are much more powerful than short-term contracts. However, Fudenberg, Holmström and Milgrom (1990) demonstrated that this is not the case. The first best can also be implemented with a sequence of short-term contracts, where the agent gets $w_t = x_t - K$ in each period, i.e. the agent is made residual claimant on the margin.

In order to get optimal insurance, the agent could follow the following self-insurance strategy:

*Starve and save until you have accumulated a wealth level of $S$. Thereafter consume the first best level of consumption and smooth income shocks by using your savings.*

Illustrate graphically.
Remarks:

- If the discount factor is close enough to 1, the first finite number of periods in which the agent starves do not count very much, and we can approach the first best arbitrarily closely.
- The problem here is again that the case where $\delta \to 1$ is not very interesting. Real life is short and agents are impatient.
The Two-Period Problem

To address the question of long-term vs. short-term contracts with impatient agents, look at a two-period moral hazard problem. Does a long-term contract strictly outperform a sequence of short-term contracts.

Lambert (1983) and Rogerson (1985) argued, that a long-term contract (which makes the wage in the second period conditional on the output of the first and the second period) is strictly better than two short-term contracts, each of which is conditional only on this periods output. However, their results rely on the assumption that the agent cannot save or get a credit.

The following result is due to Malcomson and Spinnewyn (1988) and Fudenberg, Holmström and Milgrom (1990):
Proposition 3.2

Suppose that the actions taken by the agent in period $t$ affect the output of period $t$ only, and suppose that the principal and the agent have equal and unlimited access to the capital market, then a sequence of short-term contracts is as efficient as a long-term contract.

Remarks:

1. With unequal access to the capital market, a long-term contract could be used to smooth consumption but this has nothing to do with the moral-hazard problem.

2. This result says that with moral hazard there is no benefit to long-term commitments. This is an important difference to adverse selection problems, where commitment is very important.
Robustness and Linear Contracts

Incentive schemes should be robust and give proper incentives, even if the real world is slightly different from the model that we consider. In particular, it could be the case that the action space of the agent is richer than assumed in the model.

Examples:

1. Free disposal and profit boosting
2. Intertemporal arbitrage
3. Interregional arbitrage
4. Continuous effort adjustment of the agent in the Mirrlees example.
Holmström and Milgrom (1987) consider a repeated moral hazard problem which is somewhat different from the ones we have seen so far.

- HM want to capture the idea that the agent does not choose his action at one single point in time, but that he rather adjusts his effort level more or less continuously over some time period.
- Thus, they keep the total time interval fixed, but subdivide it into more and more periods, so that the agent can choose his action more and more frequently.
- In the limit, the agent controls the drift rate of a multidimensional Brownian motion.
This model is very important:

- It shows that a linear incentive scheme is optimal in a natural class of problems.
- It offers a framework, in which it is very easy to compute the optimal incentive scheme.
- The model can be used as a building block in more complicated applied moral hazard problems.

The exposition is based on Schmidt (1996) and Hellwig and Schmidt (2002).
Holmström and Milgrom impose the following assumptions:

1. The agent consumes only at the end of the total time interval. Thus, there is no scope for consumption smoothing or self-insurance, and it is not possible to implement the first best as the number of periods goes to infinity.

2. The agent (and the principal) has constant absolute risk aversion. Thus, the agent’s risk preferences are unaffected by the history of past performance.

3. The technology is also stationary and not affected by the history of past performance.
The One-Period Problem

The agent chooses a probability vector \( p = (p_0, \ldots, p_N) \in P \), which generates a probability distribution over possible profit levels \( \tilde{\pi} \in \lbrace \pi_0, \ldots, \pi_N \rbrace \). \( P \) is the \( N \) dimensional simplex. The agent’s cost function is \( c(p) \) which is convex in \( p \).

Proposition 3.3

Suppose the principal wants to implement an action \( p \) in the interior of \( P \), such that the agent’s expected utility corresponds to the certainty equivalent \( w \). If \( p \) is implementable, then there exists a unique incentive scheme that implements \( p \):

\[
s_i = w + c(p) - \frac{1}{r} \ln \left( 1 - rc_i + r \sum_{j=0}^{N} p_j c_j \right)
\]

for \( i = 0, \ldots, N \).
The One-Period Problem

Proof: For a given incentive scheme \( s = \{ s_0, s_1, \ldots, s_N \} \), the agent’s maximization problem is:

\[
\max_{p_0, \ldots, p_N} - \sum_{i=0}^{N} p_i e^{-r(s_i - c(p))}
\]

subject to \( \sum_{i=0}^{N} p_i = 1 \). The Lagrange function of this problem is:

\[
L = - \sum_{i=0}^{N} p_i e^{-r(s_i - c(p))} - \lambda \left[ \sum_{i=0}^{N} p_i - 1 \right]
\]

The first order conditions require:

\[
\frac{dL}{dp_j} = -e^{-r(s_j - c(p))} - \sum_{i=0}^{N} p_i r c_j(p) e^{-r(s_i - c(p))} - \lambda = 0.
\]
The One-Period Problem

Multiply this with $p_j$ and take the sum over all $j$:

$$- \sum_{j=0}^{N} p_j e^{-r(s_j - c(p))} - r \sum_{j=0}^{N} p_j c_j(p) \sum_{i=0}^{N} p_i e^{-r(s_i - c(p))} = \lambda .$$

Note that the agent will be hold down to his reservation utility, so it must be the case that:

$$- \sum_{i=0}^{N} p_i e^{-r(s_i - c(p))} = U_A(w) = - e^{-rw}$$

Hence, we can write:

$$U_A(w) + rU_A(w) \sum_{j=0}^{N} p_j c_j(p) = \lambda .$$
Substituting this expression in the first order condition, we get:

\[
rc_j(p)U_A(w) - e^{-r(s_j-c(p))} - rU_A(w) \sum_{j=0}^{N} p_j c_j(p) - U_A(w) = 0
\]

which is equivalent to:

\[
e^{-r(s_j-c(p))} = e^{-rw} \left( 1 - rc_j(p) + r \sum_{j=0}^{N} p_j c_j(p) \right)
\]

Taking the logarithm on both sides yields:

\[
-rs_j + rc(p) = -rw + \ln \left( 1 - rc_j(p) + r \sum_{j=0}^{N} p_j c_j(p) \right)
\]
The One-Period Problem

or, equivalently:

\[ s_j = w + c(p) - \frac{1}{r} \ln \left( 1 - rc_j(p) + r \sum_{j=0}^{N} p_j c_j(p) \right) \]

Q.E.D.

Remarks:

1. This result is remarkable: If an action can be implemented, then there is a unique incentive scheme implementing it!
   
   In the models we considered so far there existed many incentive schemes implementing a certain action. This is why we had to look for the cost minimizing such incentive scheme which is not necessary here.

2. Note that the agent does not choose an action, but the probability distribution over outcomes directly.
3. In a model where the agent chooses a one-dimensional effort level that gives rise to a probability distribution $p$, the set of feasible probability distributions is a one-dimensional curve in $P$.

4. If the agent chooses $p$ directly out of $P$, his action space is much richer. He can now move freely in the $N$-dimensional simplex. Thus, the incentive compatibility constraints become more restrictive: the agent has to be deterred from any deviation in any direction, i.e., not just in one dimension but in $N$ dimensions. This is the reason, why there exists at most one incentive scheme implementing $p$. Illustrate graphically.
The Multi-Period Problem

We now consider the case where the above one-period problem is repeated \( m \) times.

- Suppose that in the one period problem it is optimal for the principal to implement \( p^* \) by using \( s(p^*) \).
- Let \( \tilde{A}^r_j \in \{0, 1\} \) be a random variable that is equal to 1 if \( \tilde{\pi}^r = \pi_j \) and 0 otherwise.
Proposition 3.4

The optimal incentive scheme for the m-period problem is the m-fold repetition of the optimal incentive scheme of the one-period problem. Hence, the principal wants to implement \( \bar{p}^* \) in every period and offers the incentive scheme

\[
s = \sum_{i=0}^{n} \left( s_i(\bar{p}^*) \left( \sum_{\tau=1}^{m} \tilde{A}_{\tau}^i \right) \right).
\]

Note that this incentive scheme is a linear function of the number of times each outcome materialized.

**Proof sketch:** Why is this rather trivial?
We don’t want to have a repeated moral hazard model, we rather want that the agent is able to adjust his action very frequently over time. Thus, we have to increase the number of periods while holding the total time period constant. In the limit, the agent should be able to adjust his effort continuously.

How to set up such a model is not trivial. Let me illustrate the problems for the most simple case:

- Suppose that there are \( m = \frac{1}{\Delta} \) periods, each of length \( \Delta \in \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \), indexed by \( \tau \in \{1, \ldots, \frac{1}{\Delta}\} \).
- In each period \( \tau \), there is a Bernoulli random variable

\[
X^{\tau} = \begin{cases} 
+1 \cdot \Delta X & \text{“success”} \\
-1 \cdot \Delta X & \text{“failure”}
\end{cases}
\]
In each period the agent chooses the probability
\[ p^\tau = \text{Prob}(X^\tau = +1 \cdot \Delta X) \text{ at cost } c(p^\tau). \]
We know already that the incentive scheme of the agent will be linear in the number of successes and that he will take a constant action \( p^\tau = p \) over time.

Note that
\[
E(X^\tau) = (2p - 1)\Delta X \\
\text{Var}(X^\tau) = 4p(1 - p)\Delta X^2
\]

The random variable \( X(\tau) = X^1 + X^2 + \cdots + X^\tau \) follows a binomial distribution with
\[
E(X(\tau)) = \tau \cdot (2p - 1)\Delta X \\
\text{Var}(X(\tau)) = \tau \cdot 4p(1 - p)\Delta X^2
\]
Now we want to take the limit as the number of periods goes to $\infty$ and $\Delta \to 0$.

- Let $t \in [0, 1]$ be a measure of “real time”. Ignoring integer problems, if the length of a time period is $\Delta < 1$, then there are $\frac{t}{\Delta}$ periods between date 0 and date $t$.

- At the same time we have to reduce profits in each periods and adjust the effort cost of the agent. Otherwise, profits and costs could go to infinity.

- To make the one-period problem and the $m$-period problem comparable, we want that in the limit $E(X(\frac{t}{\Delta}))$ and $Var(X(\frac{t}{\Delta}))$ are linear in $t$.

To see the problems involved consider the following two (unsuccessful) attempts:
**Attempt 1:** \( \Delta X = \Delta t \to 0 \)

\[
\lim_{\Delta t = \Delta X \to 0} \text{Var} \left( X \left( \frac{t}{\Delta t} \right) \right) = \lim_{\Delta t = \Delta X \to 0} (\Delta X)^2 \cdot \frac{t}{\Delta t} \cdot 4p(1 - p)
\]
\[
= \lim_{\Delta t \to 0} \Delta t \cdot t \cdot 4p(1 - p)
\]
\[
= 0
\]

Thus, we get a degenerate stochastic process with a variance of 0 in the limit. This process is deterministic and it is trivial to implement the first best!
Attempt 2: $\Delta X = \sqrt{\Delta t} \to 0$. Now the variance does not converge to 0, but:

$$\lim_{\Delta X = \sqrt{\Delta t} \to 0} E \left( X \left( \frac{t}{\Delta t} \right) \right) = \lim_{\Delta X = \sqrt{\Delta t} \to 0} \Delta X \cdot \frac{t}{\Delta t} \cdot (2p - 1)$$

$$= \lim_{\Delta t \to 0} \frac{\sqrt{\Delta t} \cdot t}{\Delta t} \cdot (2p - 1)$$

$$= \lim_{\Delta t \to 0} \frac{t}{\sqrt{\Delta t}} \cdot (2p - 1) = \begin{cases} +\infty & \text{if } p > \frac{1}{2} \\ -\infty & \text{if } p < \frac{1}{2} \end{cases}$$

Thus, the expected value goes either to plus or minus infinity.
How to do it properly?

In order to avoid that $E(X(\frac{t}{\Delta t})) \to \infty$, we have to redefine the action of the agent as follows:

Let $\Delta X = \sqrt{\Delta t} \to 0$ and $p = \frac{1}{2} + \frac{1}{2} \mu \sqrt{\Delta t}$. Note that if $\Delta t \to 0$, $p \to \frac{1}{2}$, but $p$ goes to $\frac{1}{2}$ more slowly than $\Delta t$ goes to 0. In the following we take $\mu$ to be the action of the agent.

$$\lim_{\Delta X = \sqrt{\Delta t} \to 0} E\left(X\left(\frac{t}{\Delta t}\right)\right) = \lim_{\Delta X = \sqrt{\Delta t} \to 0} \Delta X \cdot \frac{t}{\Delta t} \cdot (2p - 1)$$

$$= \lim_{\Delta t \to 0} \frac{\sqrt{\Delta t} \cdot t}{\Delta t} \cdot (1 + \mu \sqrt{\Delta t} - 1)$$

$$= \lim_{\Delta t \to 0} \frac{t}{\Delta t} \cdot \Delta t \cdot \mu$$

$$= \mu \cdot t$$
\[
\lim_{\Delta t \to 0} \text{Var} \left( X \left( \frac{t}{\Delta t} \right) \right) = \lim_{\Delta t \to 0} (\Delta X)^2 \cdot \frac{t}{\Delta t} \cdot 4p(1 - p)
\]
\[
= \lim_{\Delta t \to 0} \frac{\Delta t \cdot t}{\Delta t} \cdot 4 \left( \frac{1}{2} + \frac{1}{2} \mu \sqrt{\Delta t} \right) \left( \frac{1}{2} - \frac{1}{2} \mu \sqrt{\Delta t} \right)
\]
\[
= \lim_{\Delta t \to 0} t \cdot 4 \left[ \frac{1}{4} - \frac{1}{4} \mu^2 \Delta t \right]
\]
\[
= t
\]

Note that

- \(E(X(\frac{t}{\Delta t}))\) and \(\text{Var}(X(\frac{t}{\Delta t}))\) are linear in \(t\),
- \(\text{Var}(X(\frac{t}{\Delta t}))\) is independent of \(\mu\).
- In each interval \([t, t']\) we have that \(X(t') - X(t) - (t' - t)\mu\) is a sum of \(\frac{t' - t}{\Delta t}\) i.i.d. random variables with finite variance and mean 0. Thus, as \(\Delta t \to 0\), this is random variable is normally distributed with mean 0 and a variance that is proportional to \((t' - t)\).
This is a driftless Brownian motion!

**Definition 3.1**

A stochastic process \([X(t), \ t \geq 0]\) is called a (driftless) Brownian motion if it satisfies the following properties:

- \(X(0) = 0\)
- The increment \(X(t + s) - X(t)\) is normally distributed with mean 0 and variance \(\sigma^2 s\).
- For any two disjunct intervals \([t_1, t_2]\), \([t_3, t_4]\) with \(t_1 < t_2 < t_3 < t_4\) it must be the case that the increments \(X(t_4) - X(t_3)\) and \(X(t_2) - X(t_1)\) are stochastically independent.

It can be shown that a Brownian motion is continuous everywhere but nowhere differentiable. Thus, it is impossible to predict the movements of a Brownian motion.
Approximation of the Brownian Motion Model

The discussion of the previous section motivates what we have to do in our multidimensional model:

- There are \( m = \frac{1}{\Delta} \) periods, each of length \( \Delta \in \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \), indexed by \( \tau \in \{1, \ldots, \frac{1}{\Delta}\} \).

- Fix a “standard” \( \hat{p} \in P, \hat{p} \gg 0 \). This “standard” corresponds to \( p = \frac{1}{2} \) in the previous section. It is going to determine the variance of the Brownian motion.

- Given this standard we can normalize profits such that

\[
\sum_{i=0}^{N} \hat{p}_i \pi_i = 0
\]
Approximation of the Brownian Motion Model

In each period profit levels are given by:

\[ \pi_i^\Delta = \pi_i \Delta^\frac{1}{2} \quad \forall i \in \{0, \ldots, N\} . \]

Note that

\[ \sum_{i=0}^{N} \hat{p}_i \pi_i^\Delta = \Delta^\frac{1}{2} \sum_{i=0}^{N} \hat{p}_i \pi_i = 0 \]

In each period the agent chooses \( \mu^\Delta = (\mu_1^\Delta, \ldots, \mu_N^\Delta) \) which determines an associated action vector \( p^\Delta(\mu^\Delta) \):

\[ p_i^\Delta(\mu^\Delta) = \hat{p}_i + \mu_i^\Delta \frac{\Delta^\frac{1}{2}}{k_i} \quad \forall i \in \{1, \ldots, N\} , \]

\[ p_0^\Delta(\mu^\Delta) = 1 - \sum_{i=1}^{N} p_i^\Delta = \hat{p}_0 - \sum_{i=1}^{N} \mu_i^\Delta \frac{\Delta^\frac{1}{2}}{k_i} \]

where \( k_i = \pi_i - \pi_0 \). 
The agent’s cost of taking action $\hat{p}^\Delta$ are given by

$$
c^\Delta(p^\Delta) \equiv \Delta \cdot c \left( \hat{p}_0 + \frac{p_0^\Delta - \hat{p}_0}{\Delta^{\frac{1}{2}}} , \ldots , \hat{p}_N + \frac{p_N^\Delta - \hat{p}_N}{\Delta^{\frac{1}{2}}} \right)
$$

$$
= \Delta c \left( \hat{p}_0 - \sum_{i=1}^{N} \frac{\mu_i^\Delta}{k_i}, \hat{p}_1 + \frac{\mu_1^\Delta}{k_1}, \ldots , \hat{p}_N + \frac{\mu_N^\Delta}{k_N} \right)
$$

$$
\equiv \Delta \hat{c}(\mu^\Delta)
$$
What is the point of this specification?

1. Suppose the agent chooses a constant vector $p^\Delta$ (or, equivalently, a constant $\mu^\Delta$) in each period. Then the expected profit over all periods is given by:

$$\frac{1}{\Delta} \sum_{i=0}^{N} p_i^\Delta \pi_i^\Delta = \frac{1}{\Delta} \sum_{i=0}^{N} (p_i^\Delta - \hat{p}_i)(\pi_i - \pi_0)\Delta^{\frac{1}{2}}$$

$$= \sum_{i=0}^{N} (\pi_i - \pi_0) \frac{p_i^\Delta - \hat{p}_i}{\Delta^{\frac{1}{2}}} = \sum_{i=1}^{N} \mu_i^\Delta.$$

Note that the expected profit is independent of the length of each period. In particular, if the agent chooses a constant $\mu^\Delta = \mu$ in each period (independent of period length), then total profit is the same for all $\Delta$. 
2. Similarly, if the agent chooses the same $\mu$ independent of period length, then his total cost are always
\[
\frac{1}{\Delta} \Delta \hat{c}(\mu) = \hat{c}(\mu).
\]
3. Thus, the one-period and the $m$-period problem are in fact comparable.

We know by Proposition 3.3 that there exists at most one incentive scheme that implements $p^\Delta(\mu^\Delta)$. Assuming that $w = 0$ and substituting $\mu^\Delta$ and $\hat{c}(\mu^\Delta)$ we have:
\[
s_i^\Delta = \Delta \hat{c}(\mu^\Delta) - \frac{1}{r} \ln \left(1 - r \hat{c}_i k_i \Delta^{\frac{1}{2}} + r \sum_{j=0}^{N} p_j^\Delta \hat{c}_j k_j \Delta^{\frac{1}{2}}\right),
\]
where $\hat{c}_i = \frac{c_i - c_0}{k_i}$ is the partial derivative of $\hat{c}$ with respect to $\mu^\Delta_i$, and $\hat{c}_0 = 0$. 
Consider now the multi-period model with $T = \frac{1}{\Delta}$ periods of length $\Delta$. We know that the only incentive scheme that implements any given sequence of actions $\mu^\Delta$ is the sum of the incentive schemes that implement these actions in the one-period problem.

If we take the limit of the multi-period model as $T \to \infty$ and $\Delta \to 0$ we get after several simplifications (Taylor expansions) and reformulations our first main result:
Theorem 3.1

Consider a sequence of discrete models with period length $\Delta$, $\Delta = 1, \frac{1}{2}, \frac{1}{3}, \ldots$. Suppose that, as $\Delta \to 0$, the time path of actions $\mu^\Delta(t)$ converges uniformly to some continuous function $\mu(t)$, $t \in [0, 1]$. As $\Delta \to 0$,

(a) the stochastic process of cumulative deviations from the mean $X^\Delta(t) = (X^\Delta_1(t), \ldots, X^\Delta_N(t))$ converges in distribution to $X(\cdot)$ which is a driftless $N$-dimensional Brownian motion with covariance matrix

$$
\Sigma = 
\begin{pmatrix}
  k_1^2 \hat{p}_1(1 - \hat{p}_1) & -k_1 k_2 \hat{p}_1 \hat{p}_2 & \cdots & -k_1 k_N \hat{p}_1 \hat{p}_N \\
-k_2 k_1 \hat{p}_2 \hat{p}_1 & k_2^2 \hat{p}_2(1 - \hat{p}_2) & \cdots & -k_2 k_N \hat{p}_2 \hat{p}_N \\
  \vdots & \vdots & \ddots & \vdots \\
-k_N k_1 \hat{p}_N \hat{p}_1 & -k_N k_2 \hat{p}_N \hat{p}_2 & \cdots & k_N^2 \hat{p}_N(1 - \hat{p}_N)
\end{pmatrix}
$$

and starting point $X(0) = 0$;

(b) the total cost to the agent converges to $\int_0^1 \hat{c}(\mu(t)) dt$;
Theorem 3.1 (cont.)

(c) the incentive payments that serve to implement $\mu^\Delta(t)$ converge in distribution to

$$\tilde{s} = \int_0^1 \hat{c}(\mu(t))dt + \int_0^1 \hat{c}'(\mu(t))dX + \frac{r}{2} \int_0^1 \hat{c}'(\mu(t)) [\hat{c}'(\mu(t))]^T dt$$

where $\hat{c}'(\cdot) = (\hat{c}_1(\cdot), \ldots, \hat{c}_N(\cdot))$.

Remarks:

1. The proof is not that difficult, but the notation and some of the computations are a bit messy. See H&S (2002).

2. The optimal incentive scheme is equivalent to the optimal incentive scheme that Holmström and Milgrom derive for the Brownian model.
3. The difference to H&M is that H&M start with the Brownian model, while H&S show how to derive this result as the limit of a sequence of discrete models. Thus, there is no discontinuity between the discrete and the continuous model. Note that this need not be the case. There are other ways how to approximate the Brownian model that are discontinuous (Mirrlees).

4. Theorem 1 considers any converging sequence of $\mu^\Delta(t)$. However, we know already that in each discrete model the principal wants to implement the a constant $\mu^\Delta$ in each period. Thus, the sequence of optimal action paths converges to a constant $\mu^*$. For $\mu(t) = \mu^*$ we have:

$$\tilde{s} = w + \hat{c}(\mu) + \hat{c}'(\mu)X + \frac{r}{2} \hat{c}'(\mu)^* [\hat{c}'(\mu)]^T$$

where $X$ denotes the vector of cumulative deviations from the mean at time $t = 1$. 
5. Note that this incentive scheme is linear in $X$ and has an interesting interpretation:
   - The first two terms compensate the agent for his outside option and his disutility of effort.
   - The third term is the incentive payment. It is linear in the final outcome $X$.
   - The last term is the risk premium, that has to be paid to the agent. Note that this is a constant which is independent of the action that the principal wants to implement.

6. The “standard” $\hat{p}$ determines the variance-covariance matrix $\Sigma$. 
Linearity in Aggregates

So far, the optimal incentive scheme is linear in the “accounts”, that count the instances of the different possible profit levels. There may be many such accounts in which case the optimal contract would still look much more complicated than most contracts in the real world.

It would be much nicer to have a result where the optimal contract is linear in aggregates, such as total profits. This would obtain if there are only two possible output levels in each period.

With more than two profit levels, H&M offer a slightly different approach to get linearity in aggregates. Consider the Brownian model and assume that the principal does not observe each individual account over time, but only an aggregate (e.g. the sum) of all accounts over time. H&M show that this implies that the optimal contract must be linear in the aggregate.
Remarks:

1. This is the result that we are really after.

2. Unfortunately, however, this result cannot be derived as the limit of a sequence of discrete models. The reason is, that for any $\Delta > 0$ the principal can compute the time paths of the accounts for all possible profit levels from the observation of the time path of the aggregate profit level. Just compute $\pi^t - \pi^{t-1}$ and you get the profit level in period $t$. 
Hellwig and Schmidt (2002) consider a somewhat different model, in which the linearity in aggregates result can be approximated by a sequence of discrete models. They assume that

- the agent can destroy output unnoticed
- the principal observes total profits only at date 1, but he does not observe the time path of total profits.

They show that if $\Delta \to 0$, then the optimal contract of the Brownian model is $\epsilon$-optimal in the discrete model with period length $\Delta$ if $\Delta$ is sufficient small.

**Note:** This result nicely corresponds to the intuition provided by Holmström and Milgrom’s introduction. The linear scheme is optimal because the agent has a much richer action space than can be controlled by the principal.
Consider first the one-dimensional case where the principal observes total profits $x(t), t \in [0, 1]$, over time. We know by Holmströöm and Milgrom that the optimal incentive scheme must be linear in $x = \int_0^1 x(t)dt$:

$$s(x) = \alpha x + \beta.$$ 

This scheme implements the same action at each point in time, i.e., $\mu(t) = \mu$ for all $t \in [0, 1]$. Thus, total profits at date 1 are normally distributed with mean $\mu$ and some exogenously given variance $\sigma^2$. Finally, we assume that

$$c(\mu) = \frac{K}{2} \mu^2.$$ 

The First Best Contract offers full insurance to the agent and solves the following problem:

$$\max_{\mu} \{ E(x) - c(\mu) \} \iff \max_{\mu} \left\{ \mu - \frac{K}{2} \mu^2 \right\}$$
The Holmström-Milgrom Model in Action

FOC:

\[ 1 - K \mu = 0 \Rightarrow \mu_{FB} = \frac{1}{K} \]

Let the certainty equivalent of the agent’s reservation utility be \( w = 0 \). Then the agent’s participation constraint requires:

\[ 0 = s - \frac{K}{2} \mu_{FB}^2 \Rightarrow s = \frac{1}{2K} \cdot \]

Consider now the Second-Best Problem:

Which action is the agent going to choose if he is offered a linear incentive scheme \( s(x) = \alpha x + \beta \)? Consider first the certainty equivalent of his implied utility if he chooses action \( \mu \):

\[
EU = \int_{-\infty}^{\infty} -e^{-r[\alpha x + \beta - \frac{K}{2} \mu^2]} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = -e^{-rCE}
\]

\[
\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} -e^{-r\alpha x - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = -e^{-rCE + r\beta - \frac{K}{2} \mu^2}
\]
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Digression:

\[-r_\alpha x = \frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2 = \frac{1}{2} \frac{x^2 - 2\mu x + \mu^2 + 2r_\alpha x \sigma^2}{\sigma^2} = \frac{1}{2} \frac{x^2 - 2x(\mu - r_\alpha \sigma^2) + (\mu - r_\alpha \sigma^2)^2 + 2\mu r_\alpha \sigma^2 - r^2 \alpha^2 \sigma^4}{\sigma^2} = \frac{1}{2} \left[ \frac{x - (\mu - r_\alpha \sigma^2)}{\sigma} \right]^2 - r_\alpha \frac{2\mu - r_\alpha \sigma^2}{2} \]

If we plug this in above we get:

\[-e^{-r_\alpha \frac{2\mu - r_\alpha \sigma^2}{2}} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \left[ \frac{x - (\mu - r_\alpha \sigma^2)}{\sigma} \right]^2} dx = -e^{-r \left[ CE - \beta + \frac{K}{2} \mu^2 \right]} = 1 \]
The integral is equal to one because we are simply integrating over the full range of the density function of a normal distribution with mean \((\mu - r\alpha \sigma^2)\) and variance \(\sigma^2\). Thus:

\[-\frac{r\alpha(2\mu - r\alpha \sigma^2)}{2} = -r \left[ CE - \beta + \frac{K}{2\mu^2} \right] \]

\[\alpha \mu - \frac{r}{2}\alpha^2 \sigma^2 = CE - \beta + \frac{K}{2\mu^2} \]

Hence:

\[CE = \underbrace{\alpha \mu + \beta}_{\text{expected wage}} - \underbrace{\frac{K}{2\mu^2}}_{\text{effort cost}} - \underbrace{\frac{r}{2}\alpha^2 \sigma^2}_{\text{risk premium}}\]
Maximization of the agent’s expected utility is equivalent to maximizing the certainty equivalent of this utility. Note that this expression is concave in $\mu$ for any $(\alpha, \beta)$. Therefore, the optimal action of the agent is fully characterized by the FOC:

$$\mu = \frac{\alpha}{K}$$

If we want to induce the agent to choose the action $\mu$, we have to choose $\alpha$ such that

$$\alpha = \mu K$$

If the CE of the agent’s reservation utility is 0, the participation constraint requires:

$$0 = \mu \cdot K \mu + \beta - \frac{K}{2} \mu^2 - \frac{r}{2} \mu^2 K^2 \sigma^2$$

Therefore, $\beta$ has to be chosen such that:

$$\beta = -\frac{K}{2} \mu^2 + \frac{r}{2} \mu^2 K^2 \sigma^2$$
The principal maximizes $E(x - s(x)) = \mu - s(\mu)$. Hence, we can write his problem as:

$$\max_{\mu} \mu - \left( \mu K + K^2 \frac{\mu^2}{2} - r \frac{\mu^2 K^2 \sigma^2}{2} \right) - \beta$$

FOC requires:

$$1 - K \mu - rK^2 \sigma^2 \mu = 0$$

which implies:
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\[
\begin{align*}
\mu^* &= \frac{1}{K(1 + rK\sigma^2)} \\
\alpha^* &= \frac{1}{1 + rK\sigma^2} \\
\pi^* &= \mu^* - s(\mu^*) = \frac{1}{2K(1 + rK\sigma^2)}
\end{align*}
\]

This is a very nice and intuitive result:

1) If \( r = 0 \) or \( \sigma^2 = 0 \), then \( \alpha^* = 1, \mu^* = \frac{1}{K}, \pi^* = \frac{1}{2K} \), i.e., we get the first best.

2) The incentive intensity as expressed by \( \alpha^* \) increases if \( K \) decreases, \( \sigma^2 \) decreases, or \( r \) decreases.

3) The agent works harder (\( \mu^* \) increases), if \( K \) decreases, \( \sigma^2 \) decreases, or \( r \) decreases.

4) The agent works to little \( \mu^* < \mu^{FB} \).
Multiple Tasks

In this section we will discuss some interesting features that arise if we consider multiple-tasks principal-agent problems. Holmström and Milgrom (1991) offer a general analysis of these problems, but we will restrict attention to a few examples:
Example 1: Contracts for Teachers

Two tasks:

- $t_1$: Teach basic skills: reading, writing, basic math
- $t_2$: Promote the students creativity and social skills

The principal can observe the students’ test scores at the end of the year which give an imperfect signal about the teacher’s effort in teaching basic skills, but there is no test for creativity, i.e.:

- $x_1 = t_1 + \epsilon_1, \quad \sigma^2_1 < \infty$,
- $x_2 = t_2 + \epsilon_2, \quad \sigma^2_2 = \infty, \quad \sigma_{12} = 0$. 
Example 1: Contracts for Teachers

What is the optimal incentive scheme?

\[
\max_{t_1, t_2} B(t_1, t_2) - C(t_1, t_2) - \frac{1}{2} r \alpha_1^2 \sigma_1^2
\]

subject to:

(*) \quad C_1(\cdot) = \alpha_1 = \alpha

(**) \quad C_2(\cdot) = \alpha_2 = 0

Note that \( \alpha_2 = 0 \) must be optimal. Otherwise the principal would impose an infinitely high risk on the agent. Nevertheless we assume that there is an interior solution for the principal’s problem, i.e., \( \frac{\partial C(t)}{\partial t_2} < 0 \) for some \( t_2 \).

Interpretation?
Let \( t_1 = t_1(\alpha) \) and \( t_2 = t_2(\alpha) \). Then we can maximize with respect to \( \alpha \) rather than with respect to \( (t_1, t_2) \). FOC:

\[
B_1 \frac{\partial t_1}{\partial \alpha} + B_2 \frac{\partial t_2}{\partial \alpha} - C_1 \frac{\partial t_1}{\partial \alpha} - C_2 \frac{\partial t_2}{\partial \alpha} - r \alpha \sigma_1^2 = 0
\]
Totally differentiating (*) and (**), we get:

\[ C_{11} \frac{\partial t_1}{\partial \alpha} + C_{12} \frac{\partial t_2}{\partial \alpha} = 1 \]

\[ C_{21} \frac{\partial t_1}{\partial \alpha} + C_{22} \frac{\partial t_2}{\partial \alpha} = 0 \Rightarrow \frac{\partial t_1}{\partial \alpha} = - \frac{C_{22}}{C_{21}} \frac{\partial t_2}{\partial \alpha} \]

\[ \Rightarrow C_{11} \left( - \frac{C_{22}}{C_{21}} \frac{\partial t_2}{\partial \alpha} \right) + C_{12} \frac{\partial t_2}{\partial \alpha} = \frac{\partial t_2}{\partial \alpha} \left( C_{12} - \frac{C_{11} C_{22}}{C_{21}} \right) = 1. \]

This implies:

\[ \frac{\partial t_1}{\partial \alpha} = - \frac{C_{22}}{C_{12} - C_{11} C_{22}} \]

\[ \frac{\partial t_2}{\partial \alpha} = \frac{1}{C_{12} - \frac{C_{11} C_{22}}{C_{21}}}. \]
Example 1: Contracts for Teachers

Substituting this in the FOC we get:

\[(B_1 - C_1) \left( \frac{C_{22}}{C_{11}C_{22} - C_{12}^2} \right) + (B_2 - C_2) \left( \frac{C_{21}}{C_{12} - C_{11}C_{22}} \right) - r\alpha\sigma_1^2 = 0\]

Solving for \(\alpha\), we get:

\[\alpha = \frac{B_1 - B_2 \frac{C_{21}}{C_{22}}}{1 + r\sigma_1^2 (C_{11} - \frac{C_{12}^2}{C_{22}})}\]
Example 1: Contracts for Teachers

Interpretation:

1. Suppose that $C_{12} = 0$. Then we have $\alpha = \frac{B_1}{1 + rC_{11}\sigma_1}$. This is exactly the same expression that we got in the single task case.

2. Suppose that, starting at $C_{12} = 0$, $C_{12}$ decreases. Then the numerator increases while the denominator decreases, so $\alpha$ goes up. $C_{12} < 0$ means that the two tasks are complements: the more basic skills the teacher teaches, the easier it is to promote their creativity. In this case the teacher should get stronger incentives to promote basic skills than in the single task case.

3. Suppose that $C_{12}$ has the same absolute value as before but is positive. In this case the numerator becomes smaller, while the denominator is unchanged, so $\alpha$ decreases. $C_{12} > 0$ means that the two tasks are substitutes: the more effort the teacher spends on teaching basic skills, the more costly it becomes for him to teach creativity (e.g. because there is less time left). In this case the teacher should get weaker incentives as compared to the case where the tasks are complements.
Remarks:

1. If two tasks are complements, then giving stronger incentives for one task improves the agent’s performance on the other task, while if the tasks are substitutes the reverse holds.

2. There are two ways how to give stronger incentives for task $i$:
   - increase $\alpha_i$
   - increase $\alpha_j$, if task $j$ is a complement of task $i$, or decrease $\alpha_j$, if task $j$ is a substitute for task $i$.

3. Even if a task is not measurable, it is possible to affect the agent’s effort on this task by changing the incentives for other tasks that are complements or substitutes.
Example 1: Contracts for Teachers

Other examples of multi-task principal agent problems with a similar structure include:

- **Construction company**
  - $t_1$: Complete house in time
  - $t_2$: Do it properly

- **Worker**
  - $t_1$: produce output
  - $t_2$: take care of machine

- **Sales agent**
  - $t_1$: sell contracts
  - $t_2$: improve customer satisfaction
Example 2: When Are No Incentives Optimal?

Consider the following specification of the teacher’s problem:

- \( C(t) = C(t_1 + t_2) \), \( C''(\cdot) > 0 \).
- There exists a \( \bar{t} > 0 \), such that \( C'(t) \leq 0 \) for all \( t \leq \bar{t} \) and \( C'(\bar{t}) = 0 \).

Furthermore, we have:

- \( t_1 \) cannot be measured \( (\sigma_1^2 = \infty) \),
- \( x = t_2 + \epsilon, \quad \epsilon \sim N(0, \sigma^2) \),
- \( s(x) = \alpha x + \beta \),
- \( B(t_1, t_2) \) is strictly increasing in \( t_1 \) and \( t_2 \),
- \( B(0, t_2) = 0 \ \forall t_2 \), i.e., the non-measurable task is very important.
Example 2: When Are No Incentives Optimal?

**Proposition 3.5**

\( \alpha = 0 \) is optimal, even if the agent is risk neutral.

**Proof:** The agent maximizes:

\[
\max_{t_1, t_2} \alpha t_2 + \alpha \epsilon + \beta - C(t_1 + t_2)
\]

If \( \alpha = 0 \) the agent will choose \( t_1 + t_2 = \bar{t} \) and he is indifferent how to allocate his time between tasks. Hence, \( (\bar{t}_1, \bar{t}_2) \in \text{arg max } B(t_1, t_2) \) is optimal.

If \( \alpha > 0 \), the agent will choose \( t_1 = 0 \) and \( t_2 > \bar{t} \). Total surplus is:

\[
B(0, t_2) - C(t_2) - \frac{1}{2} \alpha^2 \sigma^2 < -C(\bar{t}) < B(\bar{t}_1, \bar{t}_2) - C(\bar{t})
\]

which cannot be optimal.
Example 2: When Are No Incentives Optimal?

If $\alpha < 0$, the agent will choose $t_1 = \tilde{t}$ and $t_2 = 0$. But, by definition, we have $B(\tilde{t}_1, \tilde{t}_2) > B(\tilde{t}, 0)$. The agent’s cost are the same as by $\alpha = 0$ but he must bear some additional risk. Hence, this cannot be optimal either. \textit{Q.E.D.}

Note, that in this example it is not possible to write a contract on $B(\cdot)$. Hence, even if the agent is risk neutral, it is not possible to implement the first best, because we cannot make the agent residual claimant on social surplus.
Example 3: Incentives between Firms vs. Incentives within Firms

- $t_1$: produce output,
- $t_2$: maintain machine,
- $B(t_1, t_2)$,
- $C(t_1, t_2) = C(t_1 + t_2)$.

An independent contractor typically owns the machine he is working with, while an employee typically does not own “his” machine. Why? Suppose that the owner of the machine benefits from the increase in value that is due to proper maintenance, which is not possible for an employee (e.g. because the value of the machine is not verifiable, no second hand market).
Proposition 3.6

Suppose that maintenance is sufficiently important. In this case the optimal incentive scheme pays a fixed wage to the employee ($\alpha = 0$). The optimal contract for an independent contractor has $\alpha > 0$. Both types of contracts may be optimal (depending on parameters). If $\alpha = 0$ is optimal given $(r, \sigma_1^2, \sigma_2^2)$, then this type of contract is also optimal for higher values of these parameters.


The intuition is simple: If the agent does not own his machine, then the smallest incentives to produce more output will result in zero maintenance. Thus, if maintenance is sufficiently important, it is better to give no incentives at all. However, if the agent owns his machine, he will maintain it even if he gets very strong incentives to produce output. The disadvantage of having an independent contractor is that it is impossible to insure him against fluctuations of the value of his machine.
Empirical Evidence

1) Anderson and Schmittlein (1988): independent sales agents get much stronger incentives to sell than employed sales agents. The difference is that the independent ones typically own the “client list” and have an incentive to invest in customer relations, while the employed sales agents don’t. Anderson and Schmittlein show that a given sales agent is more likely to be an employee the more difficult it is to measure his performance. Furthermore, the commission rate for independent sales agents is much higher.

2) Krueger (1991) shows that 30% of all McDonalds Restaurants are owned by McDonalds while 70% are franchises. A franchised restaurant owner gets 90% of marginal revenues, while an employed restaurant manager gets no financial incentives.