4. Adverse Selection

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Basic Readings

Textbooks:

- Bolton and Dewatripont (2005), Chapters 2 and 9
- Fudenberg and Tirole (1991, Chapter 7)
- Laffont and Martimort (2002), Chapter 2
- Schmidt (1995), Chapter 4

Papers:

- Baron and Myerson (1982)
- Laffont and Tirole (1988)

Introduction

- Definition of adverse selection?
- Adverse selection in markets and in bilateral contracts.
- Examples for adverse selection?
- Why is it important that the uninformed party makes a take-it-or-leave-it offer?
- Is this assumption a restriction or without loss of generality?

The Tow-Types Case

Price discrimination: A monopolist does not know the type of his customer(s):

- c constant marginal cost of production
- q quantity of production
- θ type of the customer, $\theta \in \{\underline{\theta}, \overline{\theta}\}$
- p probability that the monopolist assigns to the event that the customer is of type $\underline{\theta}$.
- t total payment of the customer to the monopolist.

The customer knows his own type, the monopolist knows only the ex ante probabilities $(p, 1-p)$ for each type.

The Two-Types Case

We will assume here that there is only one customer. Alternatively, we could assume that there is a continuum of customers with mass 1, the fraction p of which is of type $\underline{\theta}$. The formal analysis is the same.

Payoff functions:

$$U(q,t,\theta) = \theta u(q) - t$$

$$\Pi(q,t) = t - c \cdot q$$

Both parties are risk neutral. If the consumer does not consume anything, he gets a reservation utility of 0.

The First Best Allocation

If there was no informational asymmetry, the principal would offer a contract that solves the following problem:

$$\max_{q,t} t - cq$$

subject to

$$\text{(PC)} \qquad \theta u(q) - t \geq 0 \, .$$

In the optimal solution (PC) must hold with equality. (Why?) Thus, the first best allocation is characterized by

$$t = \theta u(q)$$

and

$$\theta u'(q) = c \, .$$

The First Best Allocation

Remarks:

1. The monopolist extracts all the rents from his customer
2. He chooses a production level such that marginal utility (or rather the MRS between this good and the expenditures for all other goods) equals marginal cost.
3. t and q are both functions of θ. This can also be written as an non-linear pricing scheme $t(q)$ (second degree price discrimination).

The Revelation Principle

What kind of contract should the principal offer if he does not know the agent's type?

There are infinitely many types of contracts ("mechanisms") that could be offered:

- a non-linear pricing scheme $t(q)$
- a revelation mechanism $\{t(\hat{\theta}), q(\hat{\theta})\}$
- a lottery: the customer buys lottery tickets at certain prices that determine his probability of "winning" a certain amount q.
- a multi-stage contract: the agent first has to pay an entrance fee, then some multi-stage game between the principal and the agent is played, the outcome of which determines the allocation
- etc.

The Revelation Principle

Thus, the problem is that the set of contracts over which the principal wants to maximize his utility is not well defined, and if we wanted to define it, if would be an incredibly complex object.

However, the **Revelation Principle** tells us, that any allocation that can be implemented at all, can also be implemented by using a direct revelation mechanism. Therefore, without loss of generality, we can restrict attention to direct revelation mechanisms which is a much simpler set of objects.

The **Revelation Principle** is extremely important. To fully understand it, we need a few definitions:

Definition 1

An **allocation**, $(q(\theta), t(\theta))$, is a function that assigns to each possible type of the agent a consumption quantity q and a payment t.

The Revelation Principle

Definition 2

A **mechanism** is a game form, i.e. a set of strategies for the agent and a function $(q(s), t(s))$ that assigns to each possible strategy of the agent an outcome (q, t).

What is the difference between a game form and a game?

Definition 3

A **revelation mechanism** (or direct mechanism) $(q(\hat{\theta}), t(\hat{\theta}))$ is a mechanism, in which the agent is asked to announce his type, i.e., the agent's strategy space S is the set of possible types Θ. Furthermore, a revelation mechanism has to be **incentive compatible**, i.e., for each possible type of the agent it has to be optimal to announce his type truthfully ($\hat{\theta} = \theta$).

Definition 4

A mechanism $\{S, (q(s), t(s))\}$ **implements** an allocation $(q(\theta), t(\theta))$ if and only if for every possible type $\theta \in \Theta$ there exists an optimal strategy $s^*(\theta)$, such that

$$(q(s^*(\theta)), t(s^*(\theta))) = (q(\theta), t(\theta))$$

The Revelation Principle

Proposition 4.1 (Revelation Principle)

The allocation function $(q(\theta), t(\theta))$ *can be implemented by some (arbitrarily complicated) mechanism if and only if it can also be implemented by a (direct) revelation mechanism.*

Proof: Let $(q(\theta), t(\theta))$ be an allocation that can be implemented by a mechanism $\{S, (q(s), t(s))\}$. Then there exists for each type $\theta \in \Theta$ an optimal strategy $s^*(\theta)$, such that

$$(q(s^*(\theta)), t(s^*(\theta))) = (q(\theta), t(\theta))$$

Let us now construct another mechanism $\left\{\Theta, (q(\hat{\theta}), t(\hat{\theta}))\right\}$ as follows:

> *If the agent claims to be type* $\hat{\theta}$*, then the allocation* $(q(s^*(\hat{\theta})), t(s^*(\hat{\theta})))$ *will be implemented.*

Note that if each type tells the truth ($\hat{\theta} = \theta$), then this direct mechanism implements exactly the same allocation as the indirect mechanism we started

The Revelation Principle

with. It remains to be shown that truthtelling is indeed optimal for each type of agent:
Note that $s^*(\theta)$ is an optimal strategy in the original mechanism. Therefore we have:

$$\theta u(q(s^*(\theta))) - t(s^*(\theta)) \;\geq\; \theta u(q(s))) - t(s) \quad \forall\, s \in S$$

In particular:

$$\theta u(q(s^*(\theta))) - t(s^*(\theta)) \;\geq\; \theta u(q(s^*(\tilde{\theta}))) - t(s^*(\tilde{\theta})) \quad \forall\, \tilde{\theta} \in \Theta$$

By the definition of our direct mechanism it follows that:

$$\theta u(q(\theta)) - t(\theta) \;\geq\; \theta u(q(\tilde{\theta})) - t(\tilde{\theta}) \quad \forall\, \tilde{\theta} \in \Theta$$

But this means that it is indeed optimal for each type $\theta \in \Theta$ to announce θ truthfully. *Q.E.D.*

The Revelation Principle

Remarks:

1. We have proved the revelation principle in the specific context of the price discrimination problem. However, it is straightforward to generalize this result to any other problem with adverse selection.
2. There are different types of implementation, corresponding to different notions of equilibrium, e.g.:
   - dominant strategy implementation (equilibrium in dominated strategies)
   - full implementation (unique Nash equilibrium)
   - truthful implementation (truthtelling is just one of potentially many Nash equilibria)
   - Bayesian Nash implementation
   - subgame perfect implementation (truthtelling is a unique, subgame perfect equilibrium)
   - virtual implementation
   - etc.

   Most of these concepts coincide if there is only one agent. Why? However, with multiple agents these concepts are quite different. See Moore (1992) for a brilliant survey of this literature.

The Revelation Principle

3. The revelation principle requires that the principal can fully commit to the terms of the contract. If this is not the case, an indirect mechanism, which allows for some commitment, may strictly outperform any direct revelation mechanism, which allows for no commitment.

How to Solve the Tow-Types Case?

Let $\overline{q} = q(\overline{\theta})$, $\underline{q} = q(\underline{\theta})$, etc. The Second Best Problem can be written as the following optimization problem:

$$\max_{\underline{q},\overline{q},\underline{t},\overline{t}} \; p(\underline{t} - c\underline{q}) + (1-p)(\overline{t} - c\overline{q})$$

subject to:

(IC1) $$\underline{\theta} u(\underline{q}) - \underline{t} \; \geq \; \underline{\theta} u(\overline{q}) - \overline{t} \; ,$$

(IC2) $$\overline{\theta} u(\overline{q}) - \overline{t} \; \geq \; \overline{\theta} u(\underline{q}) - \underline{t} \; ,$$

(PC1) $$\underline{\theta} u(\underline{q}) - \underline{t} \; \geq \; 0 \; ,$$

(PC2) $$\overline{\theta} u(\overline{q}) - \overline{t} \; \geq \; 0 \; ,$$

How to Solve the Two-Types Case

We will solve this problem in a sequence of steps:

Step 1: (PC2) is redundant and can be ignored:

$$\begin{aligned} \overline{\theta}u(\overline{q}) - \overline{t} &\geq \overline{\theta}u(\underline{q}) - \underline{t} && (1) \\ &\geq \underline{\theta}u(\underline{q}) - \underline{t} && (2) \\ &\geq 0 && (3) \end{aligned}$$

(1) is (IC2), (2) is implied by $\overline{\theta} > \underline{\theta}$ and (3) holds by (PC1).

Step 2: (PC1) must be binding in the optimal solution. If this was not the case, then it would be possible to increase $\underline{t}$ and $\overline{t}$ by $\epsilon > 0$ without violating any constraint. This would further increase the principal's payoff, a contradiction.

How to Solve the Two-Types Case

Step 3: (IC2) must be binding in the optimal solution. If this was not the case, the principal could increase $\bar{t}$ without violating any other constraint, a contradiction. To see this, illustrate the indifference curves of the different types graphically in the q, t space:

$$\theta u(q) - t = const \quad \Rightarrow \quad t = \theta u(q) - const$$

The indifference curves are concave ($u''(\cdot) < 0$), and for each q $I(\bar{\theta})$ is steeper than $I(\underline{\theta})$. This implies that the indifference curves of the different types can cross only once. This is the famous **"single crossing property"**. Without this property, adverse selection problems are very hard to solve. Note that $I(\underline{\theta})$ must pass through the origin of the diagram, because we know already that (PC1) must be binding at the optimum.

How to Solve the Two-Types Case

Let point A represent the contract that is supposed to be chosen by type $\underline{\theta}$. Consider the indifference curve $I(\overline{\theta})$ through this point.

- The contract for type $\overline{\theta}$ (point B) cannot be to the left/above of $I(\overline{\theta})$, otherwise type $\overline{\theta}$ would choose contract A.
- It cannot be to right/below $I(\underline{\theta})$ either, otherwise type $\underline{\theta}$ would choose contract B.
- Hence, $\underline{q} \leq \overline{q}$. This **monotonicity condition** is a direct implication of incentive compatibility.

Suppose, B is strictly below $I(\overline{\theta})$. Then we could increase $\bar{t}$ a little bit without violating the incentive compatibility or participation constraint for type $\underline{\theta}$. This would improve the principal's payoff, a contradiction.

How to Solve the Two-Types Case

Step 4: If $\overline{q} > \underline{q}$, then (IC1) is not binding. This is obvious from the graphical illustration in this example. In other examples it may be more difficult to show. In this case it may be better to ignore (IC1), to solve the relaxed problem, and to check afterwards whether the solution to the relaxed problem satisfies (IC1).

Step 5:

$$(PC1) \quad \Rightarrow \quad \underline{t} = \underline{\theta} u(\underline{q})$$

$$(IC2) \quad \Rightarrow \quad \overline{t} = \overline{\theta} u(\overline{q}) - \underbrace{\left[\overline{\theta} u(\underline{q}) - \underline{\theta} u(\underline{q})\right]}_{\text{information rent}}$$

Hence, our optimization problem reduces to:

$$\max_{\underline{q},\overline{q}} \; p \left[\underline{\theta} u(\underline{q}) - c\underline{q}\right] + (1-p)\left[\overline{\theta} u(\overline{q}) - \overline{\theta} u(\underline{q}) + \underline{\theta} u(\underline{q}) - c\overline{q}\right]$$

FOCs:

$$\frac{\partial}{\partial \underline{q}} = u'(\underline{q})\left[\underline{\theta} - (1-p)\overline{\theta}\right] - pc = \begin{cases} \leq 0 & \text{if } \underline{q} = 0 \\ = 0 & \text{if } \underline{q} > 0 \end{cases}$$

How to Solve the Two-Types Case

$$\frac{\partial}{\partial \overline{q}} = u'(\overline{q})(1-p)\overline{\theta} - (1-p)c = \begin{cases} \leq 0 & \text{if } \overline{q} = 0 \\ = 0 & \text{if } \overline{q} > 0 \end{cases}$$

The second order conditions are globally satisfied. To guarantee an interior solution we assume that $\underline{\theta} > (1-p)\overline{\theta}$ (otherwise $\underline{q} = 0$ is optimal), and $\lim_{q \to 0} u'(q) = \infty$ (otherwise it would not be guaranteed that $u'(q)$ is sufficiently large to guarantee an interior solution.

In an interior solution, the FOCs hold with equality:

$$\begin{aligned} \underline{\theta} u'(\underline{q}) &= \frac{c}{1 - \frac{(1-p)}{p} \frac{(\overline{\theta}-\underline{\theta})}{\underline{\theta}}} \\ \overline{\theta} u'(\overline{q}) &= c \end{aligned}$$

It is now straightforward to check algebraically that (IC1) holds.

Interpretation of the Optimal Solution

1) $\overline{q} = q^{FB}(\overline{\theta})$, i.e., "no distortion at the top". This property holds in all adverse selection models in which the single crossing property is satisfied.
2) $\underline{q} < q^{FB}(\underline{\theta})$, i.e., the "bad" type is distorted and consumes too little.
3) The intuition for these results is as follows: Note first that $q^{FB}(\underline{\theta}) < q^{FB}(\overline{\theta})$. Suppose now that $\overline{q} \neq q^{FB}(\overline{\theta})$. The we can move along the indifference curve of type $\overline{\theta}$ to $q^{FB}(\overline{\theta})$ without violating any constraint. This leaves both types of agent indifferent and increases the principal's payoff.

Interpretation

Why is $\underline{q} < q^{FB}(\underline{\theta})$?

- The principal could implement the first best and offer a contract A’ to the bad customer which induces him to consume efficiently.
- However, in this case he would have to offer the contract B’ to the good customer in order to prevent him from choosing A’ as well. This means that the good type has to get a higher **information rent**.
- Suppose the principal reduces $\underline{q}$ starting from $q^{FB}(\underline{\theta})$ a little bit along the indifference curve $I(\underline{\theta})$. This does not affect the agent and results in a second order welfare loss. Hence, the principals loss is also of second order.
- On the other hand he can now increase $\bar{t}$ which yields an additional first order profit.
- Hence, it is always optimal for the principal to distort the quantity of the bad type a little bit in order to reduce the rent that has to be left to the good type.

Interpretation

4) The degree of the distortion depends on the relative likelihood of facing the bad customer. If his probability is close to 1, then the distortion will be close to 0. In this case the expected information rent is very small anyway. On the other hand, the smaller p, the larger is the optimal distortion. There exists an $\underline{p}$, such that $\underline{q} = 0$ if $p < \underline{p}$. If $\underline{q} = 0$, the principal does not have to pay any information rent to the agent:

$$\bar{t} = \bar{\theta}u(\bar{q}) - \underbrace{\left[\bar{\theta}u(\underline{q}) - \underline{\theta}u(\underline{q})\right]}_{=0} .$$

5) In a standard adverse selection problem, there is a **trade-off between achieving allocative efficiency and minimizing the agent's rent**.

A Continuous Type Model

$V = v(x, \theta) - t$	principal's payoff function
$U = u(x, \theta) + t$	agent's payoff function
$x \in [0, \overline{x}]$	agent's action, observable and verifiable
$t \in \Re$	transfer from P to A (reverse notation!)
$\theta \in [0, 1]$	agent's type (private information)
$F(\theta)$	probability distribution over θ
$f(\theta)$	density of θ, $f(\theta) > 0 \ \forall \ \theta \in [0, 1]$.

Remarks:

1. Both payoff functions are linear in t, i.e., both players are risk neutral.
2. Specification is quite general. Examples:
   - procurement problems (e.g. Laffont and Tirole, 1993)
   - price regulation of a monopolist with unknown cost (e.g. Baron and Myerson, 1982)
   - monopolistic price discrimination (e.g. Maskin and Riley, 1984)

A Continuum of Types

- optimal taxation (e.g. Mirrlees, 1972)

Assumption 4.1

A1: $\frac{\partial u(x,\theta)}{\partial \theta} > 0$.

A2: *"Single Crossing Property":* $\frac{\partial^2 u(x,\theta)}{\partial x \partial \theta} > 0$.

A3: $u(x,\theta)$ *and* $v(x,\theta)$ *are both concave in* x*, i.e.,* $\frac{\partial^2 u(x,\theta)}{\partial x^2} < 0$, $\frac{\partial^2 v(x,\theta)}{\partial x^2} < 0$.

A4: $\frac{\partial^2 v(x,\theta)}{\partial x \partial \theta} \geq 0$, $\frac{\partial^3 u(x,\theta)}{\partial x^2 \partial \theta} \geq 0$, $\frac{\partial^3 u(x,\theta)}{\partial x \partial \theta^2} \leq 0$.

A5: *"Monotone Hazard Rate":* $\frac{\partial}{\partial \theta}\left(\frac{f(\theta)}{1-F(\theta)}\right) \geq 0$.

A6: *For any* $\theta \in [0,1]$ $\exists\, x < \overline{x}$*, such that*
$x \in \arg\max_x \{v(x,\theta) + u(x,\theta)\}$.

A Continuum of Types

Remarks:

1. A1 requires that the types can be ordered in the unit interval such that $u(\cdot)$ is monotonic in θ. It does not matter whether it is monotonically increasing or decreasing. This assumption restricts us to one-dimensional type spaces.
2. A2 is very important. It requires that utility and marginal utility go in the same direction as θ increases. Plausible in many cases, but not always.
3. A3 is necessary to make sure that the optimization problem is concave.
4. A4 is ugly because it is an assumption on third derivatives. Purely technical assumption with no natural economic interpretation. However, these assumptions are required to guarantee that the second order conditions are always satisfied.
5. A5 is very controversial. We will see later what happens if this assumptions does not hold.
6. A6 guarantees that the first best problem has a solution.

The First Best:

If the principal can observe θ he will offer a contract which is a solution to the following problem:

$$\max_{x(\theta),t(\theta)} \{v(x(\theta),\theta) - t(\theta)\}$$

subject to $u(x(\theta),\theta) + t(\theta) \geq 0$. The first best action is fully characterized by:

$$\frac{\partial v(x^{FB}(\theta),\theta)}{\partial x} + \frac{\partial u(x^{FB}(\theta),\theta)}{\partial x} = 0 ,$$

and $t^{FB}(\theta) = -u(x^{FB}(\theta),\theta)$.

The Second Best:

By the revelation principle, we can restrict attention to direct revelation mechanisms. Thus, the principal has to solve the following problem:

$$\max_{x(\theta),t(\theta)} \int_0^1 [v(x(\theta),\theta) - t(\theta)]\, dF(\theta)$$

subject to:

$$u(x(\theta),\theta) + t(\theta) \;\geq\; u(x(\hat{\theta}),\theta) + t(\hat{\theta}) \;\; \forall \theta, \hat{\theta} \in [0,1] \qquad \text{(IC)}$$

$$u(x(\theta),\theta) + t(\theta) \;\geq\; 0 \;\; \forall \theta \in [0,1] \qquad \text{(PC)}$$

We proceed in two steps: First, we characterize the set of all allocations $\{x(\theta), t(\theta)\}$ that are implementable. Then we ask, which of these feasible allocations maximizes the utility of the principal.

Second Best

Definition 5

An allocation $\{x(\theta), t(\theta)\}$ is implementable if and only if it satisfies (IC) for all $\theta, \hat{\theta} \in [0,1]$.

Proposition 4.2

An allocation $\{x(\theta), t(\theta)\}$ is implementable if and only if

$$\frac{dx(\theta)}{d\theta} \geq 0 \quad \forall \theta \in [0,1] \tag{4}$$

$$\frac{\partial u(x(\theta),\theta)}{\partial x} \cdot \frac{dx(\theta)}{d\theta} + \frac{dt(\theta)}{d\theta} = 0 \quad \forall \theta \in [0,1] \tag{5}$$

Proof: "$\Rightarrow$" First we show that (IC) implies conditions (4) and (5). FOCs of the agent's maximization problem:

Second Best

$$\frac{\partial u(x(\hat{\theta}),\theta)}{\partial x} \cdot \frac{dx(\hat{\theta})}{d\hat{\theta}} + \frac{dt(\hat{\theta})}{d\hat{\theta}} = 0 \tag{6}$$

$$\frac{\partial^2 u(x(\hat{\theta}),\theta)}{\partial x^2} \left(\frac{dx(\hat{\theta})}{d\hat{\theta}}\right)^2 + \frac{\partial u(x(\hat{\theta}),\theta)}{\partial x} \cdot \frac{d^2 x(\hat{\theta})}{d\hat{\theta}^2} + \frac{d^2 t(\hat{\theta})}{d\hat{\theta}^2} \leq 0 \tag{7}$$

If truthtelling is an optimal stragety, then (6) and (7) must be satisfied at $\hat{\theta} = \theta$. If we substitute θ for $\hat{\theta}$ in (6), then we get (5). Note that (5) must hold for all values of $\theta \in [0,1]$. Therefore, by differentiating (5) with respect to θ, we get:

$$\begin{aligned} \frac{\partial^2 u(x(\theta),\theta)}{\partial x^2} \left(\frac{dx(\theta)}{d\theta}\right)^2 \quad &+ \quad \frac{\partial^2 u(x(\theta),\theta)}{\partial x \partial \theta} \frac{dx(\theta)}{d\theta} \\ &+ \quad \frac{\partial u(x(\theta),\theta)}{\partial x} \frac{d^2 x(\theta)}{d\theta^2} + \frac{d^2 t}{d\theta^2} \quad = \quad 0 \end{aligned} \tag{8}$$

Using (8) and substituting θ for $\hat{\theta}$ in (7), we can write (7) as follows:

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$$\frac{\partial^2 u(x(\theta),\theta)}{\partial x \partial \theta} \frac{dx(\theta)}{d\theta} \geq 0 \tag{9}$$

The single crossing property (A2) implies that the first term is strictly positive. Hence, (9) implies (4).

"$\Leftarrow$" Now we have to show that any allocation $\{x(\theta), t(\theta)\}$ that satisfies (4) and (5) is implementable, i.e., (IC) is satisfied. Let $U(\hat{\theta},\theta) = u(x(\hat{\theta}),\theta) + t(\hat{\theta})$. Suppose there exists an $\hat{\theta}$, such that $U(\hat{\theta},\theta) > U(\theta,\theta)$. Then we have

$$\begin{aligned} U(\hat{\theta},\theta) - U(\theta,\theta) &= \int_{\theta}^{\hat{\theta}} \frac{\partial U(\tau,\theta)}{\partial \tau} d\tau && (10) \\ &= \int_{\theta}^{\hat{\theta}} \left[\frac{\partial u(x(\tau),\theta)}{\partial x} \cdot \frac{dx(\tau)}{d\tau} + \frac{dt(\tau)}{d\tau} \right] d\tau && (11) \\ &> 0\,. && (12) \end{aligned}$$

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Suppose $\hat{\theta} > \theta$. By (4) we know that $\frac{dx(\tau)}{d\tau} \geq 0$, and (A2) requires that $\frac{\partial u(x(\tau),\tau)}{\partial x} \geq \frac{\partial u(x(\tau),\theta)}{\partial x}$ (note that $\tau \geq \theta$). Hence, (5) implies

$$U(\hat{\theta},\theta) - U(\theta,\theta) \leq \int_{\theta}^{\hat{\theta}} \left[\frac{\partial u(x(\tau),\tau)}{\partial x} \frac{dx(\tau)}{d\tau} + \frac{dt(\tau)}{d\tau} \right] d\tau = 0 \,, \tag{13}$$

a contradiction. Analogously, suppose that $\hat{\theta} < \theta$. Then we have by (A2) $\frac{\partial u(x(\tau),\tau)}{\partial x} \leq \frac{\partial u(x(\tau),\theta)}{\partial x}$ (because now $\tau \leq \theta$), and we get

$$\begin{aligned} U(\hat{\theta},\theta) - U(\theta,\theta) &= -\int_{\hat{\theta}}^{\theta} \left[\frac{\partial u(x(\tau),\theta)}{\partial x} \frac{dx(\tau)}{d\tau} + \frac{dt(\tau)}{d\tau} \right] d\tau \\ &\leq -\int_{\hat{\theta}}^{\theta} \left[\frac{\partial u(x(\tau),\tau)}{\partial x} \frac{dx(\tau)}{d\tau} + \frac{dt(\tau)}{d\tau} \right] d\tau \\ &= 0 \,, \end{aligned} \tag{14}$$

again a contradiction. Hence, it is optimal to announce $\hat{\theta} = \theta$, and $\{x(\theta), t(\theta)\}$ is incentive compatible. *Q.E.D.*

Second Best

We have shown that (IC) is equivalent to (4) und (5). Therefore, we can reformulate the principal's problem as follows:

$$\max_{x(\theta),t(\theta)} \int_0^1 [v(x(\theta),\theta) - t(\theta)]\, dF(\theta)$$

subject to:

$$\frac{dx(\theta)}{d\theta} \geq 0 \qquad (4)$$

$$\frac{\partial u(x(\theta),\theta)}{\partial x}\frac{dx(\theta)}{d\theta} + \frac{dt(\theta)}{d\theta} = 0 \qquad (5)$$

$$u(x(\theta),\theta) + t(\theta) \geq 0 \qquad (PC)$$

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Let $U(\theta) \equiv U(\theta, \theta) = u(x(\theta), \theta) + t(\theta)$. Using (5) we get:

$$
\begin{aligned}
\frac{dU(\theta)}{d\theta} &= \frac{\partial u(x(\theta), \theta)}{\partial x} \frac{dx(\theta)}{d\theta} + \frac{\partial u(x(\theta), \theta)}{\partial \theta} + \frac{dt(\theta)}{d\theta} \\
&= \frac{\partial u(x(\theta), \theta)}{\partial \theta}
\end{aligned} \tag{15}
$$

Integrating both sides of this equation yields:

$$
\int_0^\theta \frac{dU(\tau)}{d\tau} d\tau = \int_0^\theta \frac{\partial u(x(\tau), \tau)}{\partial \tau} d\tau \tag{16}
$$

or:

$$
U(\theta) = U(0) + \int_0^\theta \frac{\partial u(x(\tau), \tau)}{\partial \tau} d\tau \tag{17}
$$

By Assumption 1 $\frac{\partial u(x(\tau), \tau)}{\partial \tau} \geq 0$, i.e., the agent's utility is monotonically increasing with his type. Thus, if (PC) holds for the worst type $\theta = 0$, then it must also hold for all other types as well. Because the principal wants to

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minimize the payment to the agent, (PC) must be binding for $\theta = 0$. (17) and (PC) binding for type $\theta = 0$ are equivalent to

$$U(\theta) = \int_0^\theta \frac{\partial u(x(\tau), \tau)}{\partial \tau} d\tau \tag{18}$$

$U(\theta)$ is the **information rent** that the principal has to pay to the agent of type θ in addition to his reservation utility, in order to induce him not to mimic any other type.
Because $U(\theta) = u(x(\theta), \theta) + t(\theta)$ we get:

$$t(\theta) = -u(x(\theta), \theta) + \int_0^\theta \frac{\partial u(x(\tau), \tau)}{\partial \tau} d\tau \tag{19}$$

Hence, we can rewrite the principal's problem:

$$\max_{x(\theta)} \int_0^1 \left[v(x(\theta), \theta) + u(x(\theta), \theta) - \int_0^\theta \frac{\partial u(x(\tau), \tau)}{\partial \tau} d\tau \right] dF(\theta) \tag{20}$$

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subject to:

$$\frac{dx(\theta)}{d\theta} \geq 0 \qquad (4)$$

The principal maximizes expected total surplus minus the expected information rent that has to be paid to the agent. (Compare to moral hazard, where the principal maximizes total surplus minus risk premium to the agent.) Consider the relaxed problem (without (4)) first. If the solution to the relaxed problem satisfies (4), we are done. Partial Integration of the last term in (20) yields:[1]

$$\int_0^1 \overbrace{\int_0^\theta \frac{\partial u(x(\tau),\tau)}{\partial \tau} d\tau}^{u(x)} \overbrace{f(\theta)\, d\theta}^{v'(x)}$$

$$= \left[\overbrace{\int_0^\theta \frac{\partial u(x(\tau),\tau)}{\partial \tau} d\tau}^{u(x)} \cdot \overbrace{F(\theta)}^{v(x)} \right]_0^1 - \int_0^1 \overbrace{\frac{\partial u(x(\theta),\theta)}{\partial \theta}}^{u'(x)} \overbrace{F(\theta)}^{v(x)} d\theta$$

[1] Digression: Partial Integration. Let $u, v : [a,b] \to R$ be two continuously differentiable functions. Then:

$$\int_a^b u(x)v'(x)dx = u(x)v(x)|_a^b - \int_a^b v(x)u'(x)dx \; .$$

Proof: Applying the product rule to the function $U := uv$ we get

$$U'(x) = u'(x)v(x) + u(x)v'(x)$$

Integrating up both sides from a to b, we get:

$$\int_a^b u'(x)v(x)dx + \int_a^b u(x)v'(x)dx = U(x)|_a^b = u(x)v(x)|_a^b \; .$$

Second Best

$$= \int_0^1 \frac{\partial u(x(\theta),\theta)}{\partial \theta} d\theta - \int_0^1 \frac{\partial u(x(\theta),\theta)}{d\theta} F(\theta) d\theta$$

$$= \int_0^1 \frac{\partial u(x(\theta),\theta)}{\partial \theta} \cdot \frac{[1-F(\theta)]}{f(\theta)} f(\theta) d\theta \; . \tag{21}$$

Hence, the principal maximizes

$$\int_0^1 \left[v(x(\theta),\theta) + u(x(\theta),\theta) - \left(\frac{1-F(\theta)}{f(\theta)} \right) \frac{\partial u(x(\theta),\theta)}{\partial \theta} \right] f(\theta) d\theta \; . \tag{22}$$

Let us ignore the possibility of a corner solution for a minute. Pointwise differentiation of (22) with respect to x yields:

$$\frac{\partial v(x,\theta)}{\partial x} + \frac{\partial u(x,\theta)}{\partial x} - \left(\frac{1-F(\theta)}{f(\theta)} \right) \frac{\partial^2 u(x,\theta)}{\partial x \partial \theta} = 0 \tag{23}$$

Second Best

To show that this condition is not only necessary but also sufficient, we have to check whether the principal's payoff function is globally concave in x:

$$\frac{\partial^2(22)}{\partial x^2} = \underbrace{\frac{\partial^2 v}{\partial x^2}}_{<0} + \underbrace{\frac{\partial^2 u}{\partial x^2}}_{<0} \underbrace{- \underbrace{\left(\frac{1-F(\theta)}{f(\theta)}\right)}_{\geq 0} \underbrace{\frac{\partial^3 u(x,\theta)}{\partial x^2 \partial\theta}}_{\geq 0}}_{<0} < 0 \,. \tag{24}$$

Note that we made use of $\frac{\partial^3 u(\cdot)}{\partial x^2 \partial\theta} \geq 0$ (Assumption A4) here.
Hence, (23) characterizes the solution of the relaxed problem.

Second Best

Now we have to check whether (4) is indeed satisfied. The implicit function theorem implies:

$$\frac{dx^*(\theta)}{d\theta} = -\frac{\frac{\partial(23)}{\partial\theta}}{\frac{\partial(23)}{\partial x}} \tag{25}$$

The denominator must be negative because of the second order condition (24). Thus, $\frac{dx^*(\theta)}{d\theta} \geq 0$ iff

$$\underbrace{\frac{\partial^2 v(\cdot)}{\partial x \partial\theta}}_{>0} + \underbrace{\frac{\partial^2 u(\cdot)}{\partial x \partial\theta}}_{>0} - \frac{\partial}{\partial\theta}\left(\frac{1-F(\theta)}{f(\theta)}\right)\underbrace{\frac{\partial^2 u(\cdot)}{\partial x \partial\theta}}_{>0}$$
$$-\underbrace{\left(\frac{1-F(\theta)}{f(\theta)}\right)}_{\geq 0}\underbrace{\frac{\partial^3 u(\cdot)}{\partial x \partial\theta^2}}_{\leq 0} \geq 0\ . \tag{26}$$

Note that we used here that $\frac{\partial^3 u(\cdot)}{\partial x \partial\theta^2} \leq 0$ (A4).

Second Best

Obviously, $\frac{\partial}{\partial\theta}\left(\frac{1-F(\theta)}{f(\theta)}\right) \leq 0$ is a sufficient condition for this inequality to hold. Note that this condition is equivalent to the "Monotone Hazard Rate" Assumption A5. Many standard distributions satisfy A5 (e.g. the uniform, normal, exponential, logistic, chi-square, or Laplace distribution). However, there is no economic reason why this assumption should be satisfied in our context. If this assumption does not hold, some intervals of types have to be pooled and will be offered the same contract.

Note, that if A5 holds, then $\frac{dx}{d\theta} > 0$.

Given Assumptions A1-A5 equation (23) fully characterizes $x^*(\theta)$. If we substitute $x^*(\theta)$ in equation (19), we get:

$$t^*(\theta) = -u(x^*(\theta), \theta) + \int_0^\theta \frac{\partial u(x^*(\tau), \tau)}{\partial \tau} d\tau \tag{27}$$

Interpretation of the Second Best Solution

The optimal contract is characterized by (23) and (27):

$$\frac{\partial v(x^*(\theta),\theta)}{\partial x} + \frac{\partial u(x^*(\theta),\theta)}{\partial x} - \left(\frac{1-F(\theta)}{f(\theta)}\right)\frac{\partial^2 u(x^*(\theta),\theta)}{\partial x \partial \theta} = 0 \qquad (23)$$

$$t^*(\theta) = -u(x^*(\theta),\theta) + \int_0^\theta \frac{\partial u(x^*(\tau),\tau)}{\partial \tau}\, d\tau \qquad (27)$$

Remarks:

1. $x^*(1) = x^{FB}(1)$, i.e., “no distortion at the top” (because $1 - F(1) = 0$).
2. $x^*(\theta) < x^{FB}(\theta)$ fof all $\theta < 1$.
3. The agent’s information rent $U(\theta) = \int_0^\theta \frac{\partial u(x^*(\tau),\tau)}{\partial \tau} d\tau$ is increasing in θ.
4. Trade-off between maximizing total surplus and minimizing the agent’s information rent.

Interpretation of the Second Best Solution

5. What happens if the Monotone Hazard Rate property does not hold? In this case we have to consider (4) explicitly. This is technically quite complicated (see e.g. Guesnerie and Laffont (1984) or Baron und Myerson (1982)).
   Intuition: If A5 does not hold, then $x^*(\theta)$ is not increasing everywhere. However, the solution must be monotonic. Therefore, intervals of types have to be pooled ("bunching"), i.e., all types in one interval get the same contract and choose the same action. Note that "bunching" can never occur in a neighborhood of 1, since

$$\frac{\partial}{\partial\theta}\left(\frac{f(\theta)}{1-F(\theta)}\right) = \frac{f'(\theta)\left[1-F(\theta)\right]+f(\theta)^2}{[1-F(\theta)]^2} \geq 0 \qquad (28)$$

   if and only if $f'(\theta)[1-F(\theta)]+f(\theta)^2 \geq 0$, which must be true if θ is close enough to 1.
6. It can be shown that it is never optimal to use a lottery in the contract as long as A4 holds. If A4 is violated, counter examples are possible.

Interpretation of the Second Best Solution

7. We assumed that the reservation utility of the agent is independent of his type. This is not always the case. If the reservation utility is increasing with θ, the technical problem arises that (PC) need not be binding for the worst type anymore (see Lewis und Sappington (1989)). Illustrate graphically.
8. The standard adverse selection model has the same problem as the standard moral hazard problem: The optimal contract crucially depends on the ex ante probability assessment of the principal which is not observable. Hence, it is difficult to obtain any testable predictions.
9. The model makes a clear prediction about the direction of the distortion: There will be underproduction for all types except for the best type. In principle this is emprically testable, and there are some indications that this is correct.
10. An important problem of a direct mechanims is the possibility of renegotiation. After the agent has revealed his type truthfully, both parties can be made better off by renegotiating the initial contract to the efficient quantity. However, if the agent anticipates renegotiation, his incentives to reveal his type truthfully are distorted (see Bester and Strausz (2001)).

Interpretation of the Second Best Solution

11. Usually, we do not observe direct revelation mechanisms in reality. However, if A5 holds and if $x^*(\theta)$ is strictly increasing, we can replace the revelation mechanism by an indirect mechanism of the form

$$T(x) = \begin{cases} t(\theta) & \text{if } \exists\theta\text{, such that } x = x(\theta) \\ 0 & \text{otherwise} \end{cases}$$

This type of contract has the advantage that it offers sometimes some protection against renegotiation.

Repeated Adverse Selection Problems

Suppose that the same adverse selection game is being played several times between the principal and the agent. The optimal contract depends on the commitment possibilities of the principal:

- full commitment
- no commitment
- limited commitment (no commitment not to renegotiate)

Full Commitment

Proposition 4.3

If the principal can fully commit to a long-term contract, then he will offer a contract that is the T-fold repetition of the optimal one-period contract.

Remarks:

1. The proof if simple. If another contract would do better in the T-period model then scaling down this contract by factor $\frac{1}{T}$ would also do better in the one-period problem, a contraction.
2. Repetition does not improve efficiency in an adverse selection context.
3. With full commitment the repeated problem is essentially a static problem. Everything is determined in period 1 already.
4. Note that if the principal can fully commit, the revelation principle applies. This is no longer the case if the principal cannot fully commit.

No Commitment

In many context the “no commitment” assumption is more plausible than the “full commitment” assumption:

- Present government cannot bind future governments
- Labor contracts have to be short-term contracts (no slavery)
- “Incomplete” contracts

With no commitment, the **ratchet effect** arises: Because the principal cannot commit not to exploit the information that he learns in the first period to expropriate the agent’s information rent in the second period, the agent will be very reluctant to reveal his type truthfully.

No Commitment

Laffont and Tirole (1988) consider a two-period model with a continuum of types and show that there is no separating equilibrium in which all types reveal themselves truthfully in period 1. Incentive constraints may bind in both directions. Problem: “Take-the-money-and-run strategies”.

Efficiency goes down as compared to full commitment. Why?

Examples for the ratchet effect?

Hart and Tirole (1988, 2 types) and Schmidt (1993, N types) consider T-period models. Main results:

- Complete pooling except for the last finite number of periods.
- Principal’s bargaining power is considerably reduced.

Limited Commitment

The most realistic case seems to be that the principal and the agent can write a long-term contract, but that they cannot commit not to renegotiate this contract.

Laffont and Tirole (1990) consider the case of a two-period model with two types. They show:

1. "Take-the-money-and-run strategies" are not a problem. Therefore, incentive constraints bind only in one direction.
2. Three different types of equilibria, which are difficult to characterize.
3. With a continuum of types full separation is possible but never optimal for the principal.